Problem 4.59

Suppose two spin-1/2 particles are known to be in the singlet configuration (Equation 4.176). Let $S_a^{(1)}$ be the component of the spin angular momentum of particle number 1 in the direction defined by the vector **a**. Similarly, let $S_b^{(2)}$ be the component of 2's angular momentum in the direction **b**. Show that

$$\left\langle S_a^{(1)} S_b^{(2)} \right\rangle = -\frac{\hbar^2}{4}\cos\theta, \tag{4.225}$$

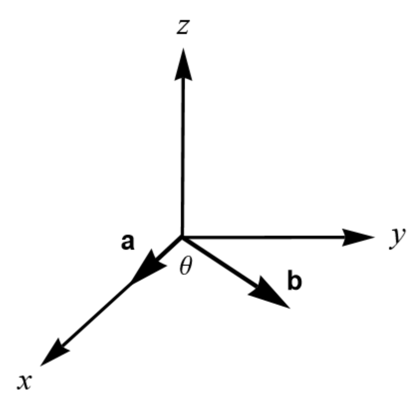
where θ is the angle between **a** and **b**.

Solution

Equation 4.225 has to hold regardless of which coordinate system is chosen.

Method 1

Choose a coordinate system where **a** lies along the x-axis and **b** lies in the xy-plane.



Evaluate the expectation value of $S_a^{(1)} S_b^{(2)}$ for the spin singlet, $|0\,0\rangle$. Let $\boldsymbol{\delta}_a$ and $\boldsymbol{\delta}_b$ be the unit vectors in the directions of **a** and **b**, respectively.

$$\begin{aligned}
\left\langle S_a^{(1)} S_b^{(2)} \right\rangle &= \left\langle 0\,0 \middle| S_a^{(1)} S_b^{(2)} \middle| 0\,0 \right\rangle \\
&= \left\langle 0\,0 \middle| (\boldsymbol{\delta}_a \cdot \mathbf{S}^{(1)})(\boldsymbol{\delta}_b \cdot \mathbf{S}^{(2)}) \middle| 0\,0 \right\rangle \\
&= \left\langle 0\,0 \middle| \left[(\cos 0) S_x^{(1)} + \left(\cos\frac{\pi}{2}\right) S_y^{(1)} + \left(\cos\frac{\pi}{2}\right) S_z^{(1)}\right] \right. \\
&\qquad \left. \left[(\cos\theta) S_x^{(2)} + \cos\left(\frac{\pi}{2} - \theta\right) S_y^{(2)} + \left(\cos\frac{\pi}{2}\right) S_z^{(2)}\right] \middle| 0\,0 \right\rangle \\
&= \left\langle 0\,0 \middle| S_x^{(1)} \left[(\cos\theta) S_x^{(2)} + (\sin\theta) S_y^{(2)}\right] \middle| 0\,0 \right\rangle \\
&= \left\langle 0\,0 \middle| S_x^{(1)} \left[(\cos\theta) S_x^{(2)} + (\sin\theta) S_y^{(2)}\right] \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \right.
\end{aligned}$$

Write the spin operators in terms of the raising and lowering operators: $S_x = \frac{1}{2}(S_+ + S_-)$ and $S_y = \frac{1}{2i}(S_+ - S_-)$.

$$\left\langle S_a^{(1)} S_b^{(2)} \right\rangle = \left\langle 0\,0 \middle| \frac{1}{2}(S_+^{(1)} + S_-^{(1)}) \left[(\cos\theta)\frac{1}{2}(S_+^{(2)} + S_-^{(2)}) + (\sin\theta)\frac{1}{2i}(S_+^{(2)} - S_-^{(2)}) \right] \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \right.$$

$$= \left\langle 0\,0 \middle| (S_+^{(1)} + S_-^{(1)}) \left[(\cos\theta - i\sin\theta) S_+^{(2)} + (\cos\theta + i\sin\theta) S_-^{(2)} \right] \frac{1}{4\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \right.$$

$$= \left\langle 0\,0 \middle| (S_+^{(1)} + S_-^{(1)})(e^{-i\theta} S_+^{(2)} + e^{i\theta} S_-^{(2)}) \frac{1}{4\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \right.$$

$$= \frac{1}{4\sqrt{2}} \left\langle 0\,0 \middle| (S_+^{(1)} + S_-^{(1)}) \left[e^{-i\theta} S_+^{(2)} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) + e^{i\theta} S_-^{(2)} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \right] \right.$$

$$= \frac{1}{4\sqrt{2}} \left\langle 0\,0 \middle| (S_+^{(1)} + S_-^{(1)}) \left[e^{-i\theta} |\uparrow\rangle \left(S_+^{(2)} |\downarrow\rangle \right) - e^{-i\theta} |\downarrow\rangle \left(S_+^{(2)} |\uparrow\rangle \right) \right. \right.$$

$$\left. + e^{i\theta} |\uparrow\rangle \left(S_-^{(2)} |\downarrow\rangle \right) - e^{i\theta} |\downarrow\rangle \left(S_-^{(2)} |\uparrow\rangle \right) \right]$$

$$= \frac{1}{4\sqrt{2}} \left\langle 0\,0 \middle| (S_+^{(1)} + S_-^{(1)}) \left[e^{-i\theta} |\uparrow\rangle (\hbar |\uparrow\rangle) - e^{-i\theta} |\downarrow\rangle (\mathbf{0}) + e^{i\theta} |\uparrow\rangle (\mathbf{0}) - e^{i\theta} |\downarrow\rangle (\hbar |\downarrow\rangle) \right] \right.$$

$$= \frac{\hbar}{4\sqrt{2}} \left\langle 0\,0 \middle| (S_+^{(1)} + S_-^{(1)}) \left(e^{-i\theta} |\uparrow\uparrow\rangle - e^{i\theta} |\downarrow\downarrow\rangle \right) \right.$$

$$= \frac{\hbar}{4\sqrt{2}} \left\langle 0\,0 \middle| \left[\left(e^{-i\theta} S_+^{(1)} |\uparrow\uparrow\rangle - e^{i\theta} S_+^{(1)} |\downarrow\downarrow\rangle \right) + \left(e^{-i\theta} S_-^{(1)} |\uparrow\uparrow\rangle - e^{i\theta} S_-^{(1)} |\downarrow\downarrow\rangle \right) \right] \right.$$

$$= \frac{\hbar}{4\sqrt{2}} \left\langle 0\,0 \middle| \left[e^{-i\theta} \left(S_+^{(1)} |\uparrow\rangle \right) |\uparrow\rangle - e^{i\theta} \left(S_+^{(1)} |\downarrow\rangle \right) |\downarrow\rangle + e^{-i\theta} \left(S_-^{(1)} |\uparrow\rangle \right) |\uparrow\rangle - e^{i\theta} \left(S_-^{(1)} |\downarrow\rangle \right) |\downarrow\rangle \right] \right.$$

$$= \frac{\hbar}{4\sqrt{2}} \left\langle 0\,0 \middle| \left[e^{-i\theta} (\mathbf{0}) |\uparrow\rangle - e^{i\theta} (\hbar |\uparrow\rangle) |\downarrow\rangle + e^{-i\theta} (\hbar |\downarrow\rangle) |\uparrow\rangle - e^{i\theta} (\mathbf{0}) |\downarrow\rangle \right] \right.$$

$$= \frac{\hbar^2}{4\sqrt{2}} \left\langle 0\,0 \middle| \left(-e^{i\theta} |\uparrow\downarrow\rangle + e^{-i\theta} |\downarrow\uparrow\rangle \right) \right.$$

$$= \frac{\hbar^2}{4\sqrt{2}} \langle 0\,0 | \left[(\cos\theta - i\sin\theta) |\downarrow\uparrow\rangle - (\cos\theta + i\sin\theta) |\uparrow\downarrow\rangle \right]$$

$$= \frac{\hbar^2}{4} \langle 0\,0 | \left[\frac{\cos\theta}{\sqrt{2}} (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) - \frac{i\sin\theta}{\sqrt{2}} (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) \right]$$

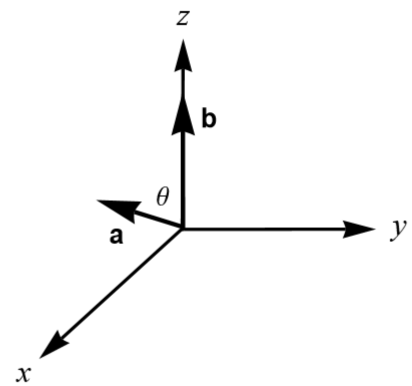
$$= \frac{\hbar^2}{4} \langle 0\,0 | \left[\cos\theta \,(-|0\,0\rangle) - i\sin\theta \,(|1\,0\rangle) \right]$$

Therefore, because the states are orthonormal,

$$\begin{aligned}\left\langle S_a^{(1)} S_b^{(2)} \right\rangle &= -\frac{\hbar^2}{4}\cos\theta\,\langle 0\,0|0\,0\rangle - \frac{i\hbar^2}{4}\sin\theta\,\langle 0\,0|1\,0\rangle \\ &= -\frac{\hbar^2}{4}\cos\theta\,(1) - \frac{i\hbar^2}{4}\sin\theta\,(0) \\ &= -\frac{\hbar^2}{4}\cos\theta.\end{aligned}$$

Method 2

Choose a coordinate system where **b** lies along the z-axis and **a** lies in the xz-plane.



Evaluate the expectation value of $S_a^{(1)}S_b^{(2)}$ for the spin singlet, $|0\,0\rangle$. Let $\boldsymbol{\delta}_a$ and $\boldsymbol{\delta}_b$ be the unit vectors in the directions of **a** and **b**, respectively.

$$\begin{aligned}
\left\langle S_a^{(1)}S_b^{(2)}\right\rangle &= \left\langle 0\,0\middle|S_a^{(1)}S_b^{(2)}\middle|0\,0\right\rangle \\
&= \left\langle 0\,0\middle|(\boldsymbol{\delta}_a\cdot\mathbf{S}^{(1)})(\boldsymbol{\delta}_b\cdot\mathbf{S}^{(2)})\middle|0\,0\right\rangle \\
&= \left\langle 0\,0\middle|\left[\cos\left(\frac{\pi}{2}-\theta\right)S_x^{(1)}+\left(\cos\frac{\pi}{2}\right)S_y^{(1)}+(\cos\theta)S_z^{(1)}\right]\right. \\
&\qquad\qquad \left.\left[\left(\cos\frac{\pi}{2}\right)S_x^{(2)}+\left(\cos\frac{\pi}{2}\right)S_y^{(2)}+(\cos 0)S_z^{(2)}\right]\middle|0\,0\right\rangle \\
&= \left\langle 0\,0\middle|\left[(\sin\theta)S_x^{(1)}+(\cos\theta)S_z^{(1)}\right]S_z^{(2)}\middle|0\,0\right\rangle \\
&= \left\langle 0\,0\middle|\left[(\sin\theta)S_x^{(1)}+(\cos\theta)S_z^{(1)}\right]S_z^{(2)}\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle\right)\right. \\
&= \frac{1}{\sqrt{2}}\left\langle 0\,0\middle|\left[(\sin\theta)S_x^{(1)}+(\cos\theta)S_z^{(1)}\right]\left[|\uparrow\rangle\left(S_z^{(2)}|\downarrow\rangle\right)-|\downarrow\rangle\left(S_z^{(2)}|\uparrow\rangle\right)\right]\right. \\
&= \frac{1}{\sqrt{2}}\left\langle 0\,0\middle|\left[(\sin\theta)S_x^{(1)}+(\cos\theta)S_z^{(1)}\right]\left[|\uparrow\rangle\left(-\frac{\hbar}{2}|\downarrow\rangle\right)-|\downarrow\rangle\left(\frac{\hbar}{2}|\uparrow\rangle\right)\right]\right. \\
&= -\frac{\hbar}{2\sqrt{2}}\left\langle 0\,0\middle|\left[(\sin\theta)S_x^{(1)}+(\cos\theta)S_z^{(1)}\right]\left(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle\right)\right. \\
&= -\frac{\hbar}{2\sqrt{2}}\left\langle 0\,0\middle|\left[(\sin\theta)S_x^{(1)}\left(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle\right)+(\cos\theta)S_z^{(1)}\left(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle\right)\right]\right. \\
&= -\frac{\hbar}{2\sqrt{2}}\left\langle 0\,0\middle|\left[(\sin\theta)\frac{1}{2}(S_+^{(1)}+S_-^{(1)})\left(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle\right)+(\cos\theta)S_z^{(1)}\left(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle\right)\right]\right.
\end{aligned}$$

Therefore, because the states are orthonormal,

$$\begin{aligned}
\left\langle S_a^{(1)} S_b^{(2)} \right\rangle &= -\frac{\hbar}{2\sqrt{2}} \left\langle 0\,0 \right| \left[\frac{\sin\theta}{2} \left(S_+^{(1)} |\uparrow\downarrow\rangle + S_+^{(1)} |\downarrow\uparrow\rangle + S_-^{(1)} |\uparrow\downarrow\rangle + S_-^{(1)} |\downarrow\uparrow\rangle \right) \right. \\
&\qquad \left. + \cos\theta \left(S_z^{(1)} |\uparrow\downarrow\rangle + S_z^{(1)} |\downarrow\uparrow\rangle \right) \right] \\
&= -\frac{\hbar}{2\sqrt{2}} \left\langle 0\,0 \right| \left\{ \frac{\sin\theta}{2} \left[\left(S_+^{(1)} |\uparrow\rangle \right) |\downarrow\rangle + \left(S_+^{(1)} |\downarrow\rangle \right) |\uparrow\rangle + \left(S_-^{(1)} |\uparrow\rangle \right) |\downarrow\rangle + \left(S_-^{(1)} |\downarrow\rangle \right) |\uparrow\rangle \right] \right. \\
&\qquad \left. + \cos\theta \left[\left(S_z^{(1)} |\uparrow\rangle \right) |\downarrow\rangle + \left(S_z^{(1)} |\downarrow\rangle \right) |\uparrow\rangle \right] \right\} \\
&= -\frac{\hbar}{2\sqrt{2}} \left\langle 0\,0 \right| \left\{ \frac{\sin\theta}{2} \left[(\mathbf{0}) |\downarrow\rangle + (\hbar |\uparrow\rangle) |\uparrow\rangle + (\hbar |\downarrow\rangle) |\downarrow\rangle + (\mathbf{0}) |\uparrow\rangle \right] \right. \\
&\qquad \left. + \cos\theta \left[\left(\frac{\hbar}{2} |\uparrow\rangle \right) |\downarrow\rangle + \left(-\frac{\hbar}{2} |\downarrow\rangle \right) |\uparrow\rangle \right] \right\} \\
&= -\frac{\hbar^2}{4} \left\langle 0\,0 \right| \left[\frac{\sin\theta}{\sqrt{2}} (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) + \frac{\cos\theta}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \right] \\
&= -\frac{\hbar^2}{4} \left\langle 0\,0 \right| \left[\frac{\sin\theta}{\sqrt{2}} (|1\,1\rangle + |1\ -1\rangle) + \cos\theta\, (|0\,0\rangle) \right] \\
&= -\frac{\hbar^2}{4} \left[\frac{\sin\theta}{\sqrt{2}} (\langle 0\,0|1\,1\rangle + \langle 0\,0|1\ -1\rangle) + \cos\theta\, \langle 0\,0|0\,0\rangle \right] \\
&= -\frac{\hbar^2}{4} \left[\frac{\sin\theta}{\sqrt{2}} (0+0) + \cos\theta\, (1) \right] \\
&= -\frac{\hbar^2}{4} \cos\theta.
\end{aligned}$$